Notes on Linear Programming: Part XV

MINIMIZING THE NUMBER OF CARRIERS TO MEET A FIXED SCHEDULE

by

G. B. Dantzig
D. R. Fulkerson

P-569

24 August 1954

The RAND Corporation
1700 MAIN ST. · SANTA MONICA · CALIFORNIA

SUMMARY

It is shown that the problem of determining the minimum number of carriers required to meet a fixed schedule of transportation can be made into a linear programming problem. ( )

MINIMIZING THE NUMBER OF CARRIERS TO MEET A FIXED SCHEDULE

by

G. B. Dantzig
D. R. Fulkerson

1. INTRODUCTION

C. Tompkins (see [3]) has given a discrete idealization of a scheduling problem which arose in the routing of Navy fuel oil tankers. A combinatorial problem of this kind has also been discussed by J. Robinson and J. Walsh [2], together with a proposed method of computation. The algorithm they outline, however, fails to narrow the number of possibilities sufficiently to make it a feasible computational method for most problems.

In this note we show how the tanker scheduling problem can be made into a linear programming problem of transportation type, albeit large. The size of the system is mitigated somewhat by the following facts: (1) most of the variables are constrained to be zero, (2) the minimizing form is particularly simple, and (3) even a large transportation type problem having no special features can be solved by hand using the simplex algorithm [1].

2. THE PROBLEM

A rectangular array of spaces is furnished, one row for each pickup point and one column for each discharge point. In each space (i,j), $i=1,2,\ldots,m$; $j=1,2,\ldots,n$ is a sequence of numbers t_{ij}^k, $k=1,2,\ldots$ representing the times at which a tanker is to load fully at pickup point i to deliver to destination j. For example, in the array

	1	2	3
1	1, 4, 7, 10, 13	9, 15	6, 12
2	3, 6, 9, 12	7, 10, 13, 15	5, 10, 15

the sequence 3, 6, 9, 12 in box (2,1) means that at these times a tanker is to begin loading at pickup point 2 for delivery to discharge point 1. Multiple loads to go from i to j can be taken care of by repetitions in the sequence t_{ij}^k. We make the further assumption, not made in [3], that the total number of entries in the table $t = (t_{ij}^k)$ is finite.

In addition, two arrays of positive numbers a_{ij} and b_{ij} are given, where a_{ij} represents the loading-traveling time from i to j, and b_{ij} the unloading-traveling time from j to i.

The problem is to rearrange the numbers t_{ij}^k into a sequences such that

(2.1) each sequence is monotone increasing;

(2.2) if $t^{k_1}_{i_1 j_1} < t^{k_2}_{i_2 j_2}$ are consecutive numbers in any one of the s sequences, then

$$t^{k_2}_{i_2 j_2} - t^{k_1}_{i_1 j_1} \geq a_{i_1 j_1} + b_{i_2 j_1} \quad ;$$

(2.3) s is minimal.

In other words, each sequence is a schedule for one tanker, and the objective is to meet the fixed schedule given by table t with a minimum number of tankers.

If, in the example, we take

$$a_{ij} = b_{ij} = \begin{array}{|c|c|c|} \hline 2 & 3 & 2 \\ \hline 1 & 2 & 1 \\ \hline \end{array} \quad .$$

then a feasible schedule using seven tankers would be represented by the rearrangement

(1)	$t^1_{11} = 1$,	$t^1_{23} = 5$,	$t^1_{22} = 7$,	$t^5_{11} = 13$
(2)	$t^1_{21} = 3$,	$t^2_{21} = 6$,	$t^3_{21} = 9$,	$t^2_{12} = 15$
(3)	$t^2_{11} = 4$,	$t^1_{12} = 9$,	$t^4_{22} = 15$	
(4)	$t^1_{13} = 6$,	$t^2_{22} = 10$,	$t^3_{23} = 15$	
(5)	$t^3_{11} = 7$,	$t^2_{23} = 10$,	$t^3_{22} = 13$	
(6)	$t^2_{13} = 12$			
(7)	$t^4_{11} = 10$.			

3. REFORMULATION AS A PROGRAMMING PROBLEM

For convenience in exposition, we suppose that the numbers t^k_{ij}, a_{ij}, b_{ij} are positive integers. It will be clear that this is not essential to the method of solution.

First construct the table of sequences $T = (t^k_{ij} + a_{ij})$. Thus, T^k_{ij} are the times when tankers loaded at i will arrive at j. Define

$n_{\alpha i}$ = number of times $t^k_{ij} = \alpha$ occurs in row i of t;

$N_{\beta j}$ = number of times $T^k_{ij} = \beta$ occurs in column j of T;

i.e., $n_{\alpha i}$ is the number of tankers loading at i at time α and $N_{\beta j}$ is the number arriving at j at time β. Thus, $n_{\alpha i}$ is defined for $\alpha=1,2,\ldots,\max t^k_{ij}$ and $N_{\beta j}$ is defined for $\beta=1,2,\ldots,\max T^k_{ij}$.

For any schedule, denote the number of reassignments from discharge point j at time β to loading point i at time α by $x_{\alpha i\beta j}$. Then for all possible schedules, the inequalities

$$(3.1) \qquad \sum_{\alpha,i} x_{\alpha i\beta j} \leq N_{\beta j}$$

$$\sum_{\beta,j} x_{\alpha i\beta j} \leq n_{\alpha i}$$

$$(3.2) \qquad x_{\alpha i\beta j} \geq 0$$

are satisfied. In addition, it follows from (2.1) and (2.2) that

$$(3.3) \qquad b_{ij} > \alpha-\beta \text{ implies } x_{\alpha i\beta j} = 0 .$$

The system (3.1) can be made into a system of equalities which is formally of transportation type by introducing non-negative slack variables $x_{\alpha i}$, $y_{\beta j}$ and $z = \sum_{\alpha,i} \sum_{\beta j} x_{\alpha i \beta j}$. Then (3.1) may be rewritten as

$$
\begin{aligned}
(3.4)\quad & \sum_{\alpha,i} x_{\alpha i \beta j} + y_{\beta j} = N_{\beta j}, && y_{\beta j} \geq 0 \\
& \sum_{\beta,j} x_{\alpha i \beta j} + x_{\alpha i} = n_{\alpha i}, && x_{\alpha i} \geq 0 \\
& \sum_{\alpha,i} x_{\alpha i} + z = \sum_{\alpha,i} n_{\alpha i}, && z \geq 0 \\
& \sum_{\beta,j} y_{\beta j} + z = \sum_{\beta,j} N_{\beta j},
\end{aligned}
$$

and hence each schedule leads to an integral solution of (3.2) and (3.4) which satisfies condition (3.3).

Conversely, given any integral solution of (3.2), (3.3), and (3.4), a schedule can be constructed from it as follows: Each $x_{\alpha i}$ will be the number of tankers which start their individual schedules at time α from loading point i; i.e., there will be $x_{\alpha i}$ sequences in the rearrangement which have a $t^k_{ij} = \alpha$ as first member. Delete one such $t^{k_o}_{i_o j_o} = \alpha_o$ from t; let $\beta_o = T^{k_o}_{i_o j_o} - \alpha_o + a_i$

Since $N_{\beta_o j_o} > 0$, at least one of the variables $x_{\alpha i \beta_o j_o}$, $y_{\beta_o j_o}$ has a positive value. Select one such.

<u>Case 1</u>. $x_{\alpha_1 i_1 \beta_o j_o} > 0$ was selected. Then there is a $t^{k_1}_{i_1 j_1} = \alpha_1$, since $n_{\alpha_1 i_1} > 0$. Assign α_1 as second member of the sequence. Observe that by (3.3), $\alpha_1 - \beta_o \geq t_{i_1 j_o}$, hence

$$t_{i_1 j_1}^{k_1} - t_{i_0 j_0}^{k_0} = \alpha_1 - \beta_0 + a_{i_0 j_0} \geq a_{i_0 j_0} + b_{i_1 j_0}$$

and (2.2) is satisfied. Strike out α_1 from t and reduce $x_{\alpha_1 i_1 \beta_0 j_0}$, $N_{\beta_0 j_0}$, $n_{\alpha_1 i_1}$ by unity.

Case 2. $y_{\beta_0 j_0} > 0$ was selected. In this case the sequence ends with α_0. Reduce $y_{\beta_0 j_0}$, $N_{\beta_0 j_0}$ by unity.

If Case 1 obtained, let $\beta_1 = T_{i_1 j_1}^{k_1} = \alpha_1 + a_{i_1 j_1}$, and examine the values of the variables $x_{\alpha i \beta_1 j_1}$, $y_{\beta_1 j_1}$. Again one of these must be positive. Apply either Case 1 or Case 2 with α_1 playing the role of α_0. Repetition of the procedure outlined must eventually end with the selection of some $y_{\beta_k j_k} > 0$ (since by (3.3), $\alpha \leq \beta$ implies $x_{\alpha i \beta j} = 0$), thus completing one of the sequences. The others can be gotten in the same way.

Notice that while many schedules can be constructed from an integral solution of (3.2), (3.3), and (3.4), the only physical difference between two such is that there may be more than one tanker available at the same time at some discharge point, in which case they are interchangeable.

Thus, the tanker scheduling problem can be viewed as one of minimizing $\sum_{\alpha, i} x_{\alpha i}$, the number of sequences in a rearrangement (or what is the same thing, maximizing the variable z), over the set of integral solutions* of (3.2) and (3.4) in which the

---

*Such whole number solutions always exist, e.g., take all $x_{\alpha i \beta j} = 0$. This corresponds to the worst possible schedule of assigning a different tanker for each trip.

variables designated by (3.3) are fixed at zero. But it is well known [1] that the maximum of a linear form defined over all solutions of (3.2) and (3.4) is always assumed at some integral solution, and it is easy to see that this fact is not altered by imposing additional constraints of the form $x_{\alpha i \beta j} = 0$. Hence the scheduling problem can be solved by the simplex algorithm, since the nature of the algorithm is such as to obtain a required integral solution. Moreover, the algorithm is extremely simple to apply when the problem is of transportation type, as is this one.

It is obvious also that linear programming can be used to optimize schedules with respect to other costs. For example, it would be simply a matter of changing the minimizing form, holding z fixed, to find a schedule for a given number of tankers which has the least sailing time.

4. A NUMERICAL EXAMPLE

We continue with the example of §2. First form the table of arrival times

T =

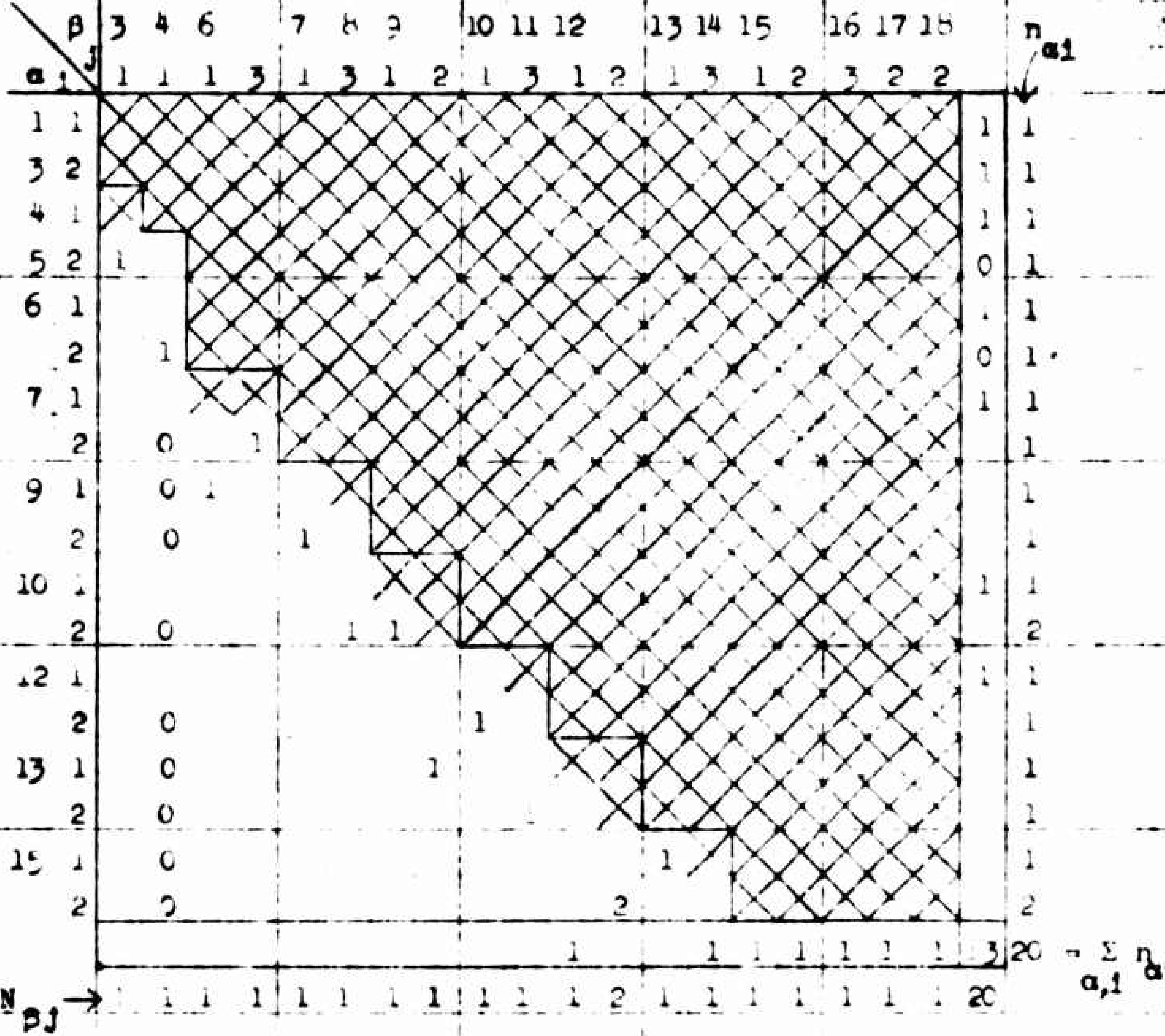
	1	2	3
1	3, 6, 9, 12, 15	12, 18	8, 14
2	4, 7, 10, 13	9, 12, 15, 17	6, 11, 16

Using this table and the one of loading times, compute all $n_{\alpha i}$

nd $N_{\beta j}$. After discarding those rows and columns having $n_{\alpha i}$ or βj zero, one is left with the transportation problem whose onstraints are indicated schematically in Fig. 1. Crossed out ells mean that the corresponding variable is constrained to be ero by (3.3). The solution shown in Fig. 1 corresponds to the chedule using seven tankers given in §2. This is a degenerate olution to the programming problem and so it is necessary in pplying the simplex algorithm to pick out other basic variables aving zero values. One way of doing this is shown in Fig. 1 by he placement of the 0's.

An optimal solution, reached after a few iterations, is hown in Fig. 2. It corresponds to the following six-tanker chedule:

(1)	$t_{11}^{1} = 1$,	$t_{23}^{1} = 5$,	$t_{22}^{1} = 7$,	$t_{11}^{5} = 13$
(2)	$t_{21}^{1} = 3$,	$t_{21}^{2} = 6$,	$t_{21}^{3} = 9$,	$t_{13}^{2} = 12$
(3)	$t_{11}^{2} = 4$,	$t_{12}^{1} = 9$,	$t_{22}^{4} = 15$	
(4)	$t_{13}^{1} = 6$,	$t_{22}^{2} = 10$,	$t_{23}^{3} = 15$	
(5)	$t_{11}^{3} = 7$,	$t_{23}^{2} = 10$,	$t_{21}^{4} = 12$,	$t_{12}^{2} = 15$
(6)	$t_{11}^{4} = 10$,	$t_{22}^{3} = 13$.		


β
j
α
i
3 4 6 7 8 9 10 11 12 13 14 15 16 17 18
$n_{\alpha i}$
$N_{\beta j}$
20 = Σ $n_{\alpha i}$
α,i

Figure

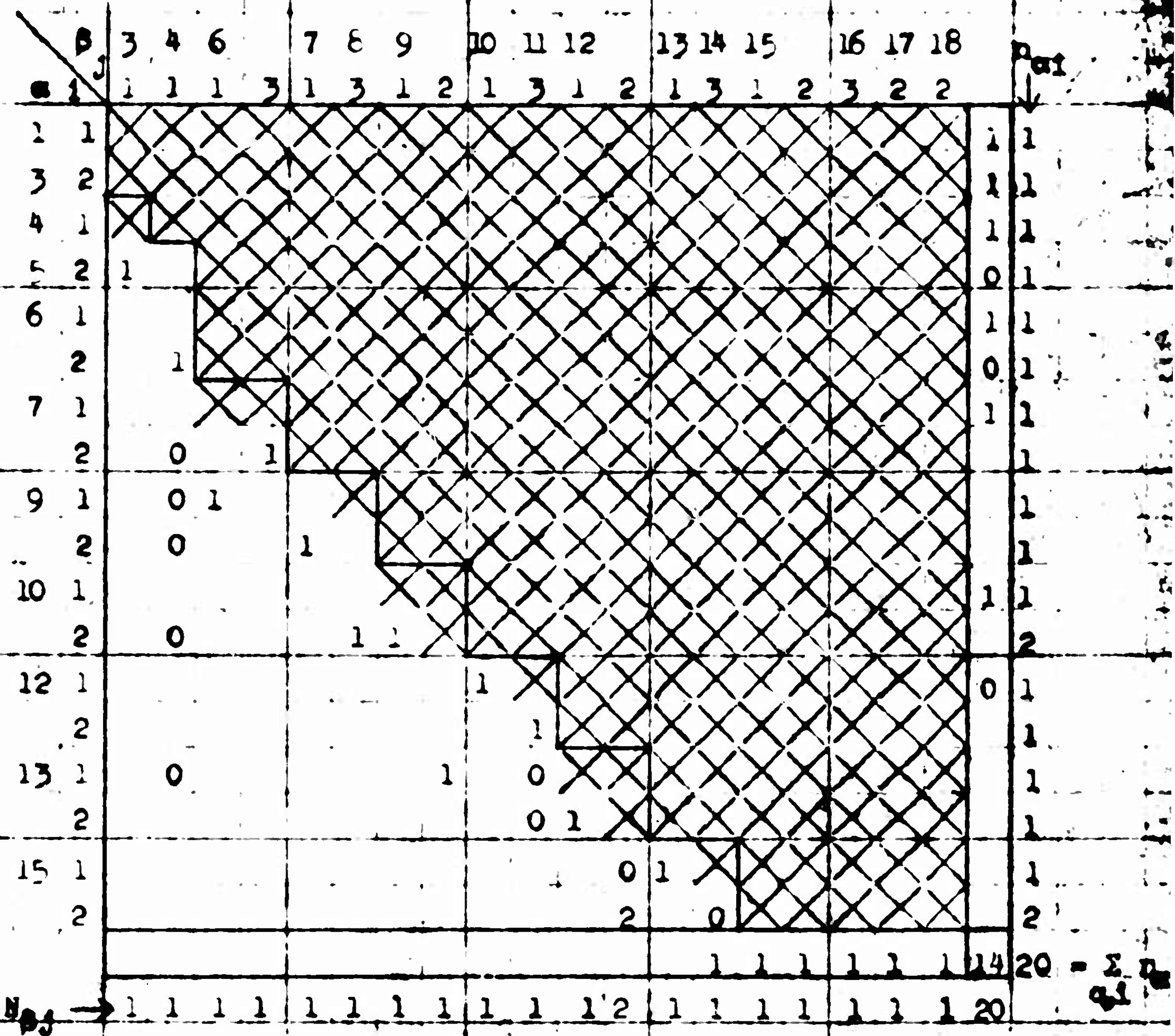


Figure 2

REFERENCES

1. G. Dantzig, "Application of the Simplex Method to a Transportation Problem," Activity Analysis of Production and Allocation, Cowles Commission Monograph No. 13, 1951.

2. J. Robinson and J. Walsh, Routing of Empties for Fixed-Schedule Transportation, The RAND Corporation, Memorandum M-406, June 12, 1950.

3. C. Tompkins, Discrete Problems and Computers, INA-53-5, November 17, 1952, pp. 14-16.